Chapter 16 Problem 81 †

Given
$T_c = -15\ °C$
$T_0 = 20\ °C$
$C = 6.5 \times 10^6\ J/K$
$R = 6.67\ mK/W = 0.00667\ K/W$

Solution
Find the time for the house to cool to $0\ °C$.

Equation 16.3 is

$$Q = mc\Delta T$$

The heat capacity given in the problem is a combination of the mass and the specific heat of the house. Therefore $C = mc$ and

$$Q = C\Delta T$$

Q is measured in Joules and we want to find the heat flow in watts. As a differential amount of heat, dQ, leaves the house, the temperature changes by a differential amount, dT. Therefore, the differential form of this equation is

$$dQ = CdT$$

If the change is occuring with respect to time, then the heat flow in watts is

$$\frac{dQ}{dt} = C\frac{dT}{dt}$$

Next take equation 16.5

$$H = -kA\frac{\Delta T}{\Delta x}$$

Thermal resistance is defined as $R = \Delta x/(kA)$. This equation becomes

$$H = -\frac{\Delta T}{R}$$

Replace the difference in temperature with the current temperature, T, and the outside temperature, T_c, and we have

$$H = -\frac{T - T_c}{R}$$

Remember that $H = dQ/dt$; therefore,

$$-\frac{T - T_c}{R} = H = \frac{dQ}{dt} = C\frac{dT}{dt}$$

or

$$-\frac{(T - T_c)}{CR} = \frac{dT}{dt} \tag{1}$$

†Problem from Essential University Physics, Wolfson

This is a first order differential equation and can be solved by u-substitution. Let

$$u = T - T_c$$

$$du = d(T - T_c) = dT$$

Substitute into equation 1

$$-\frac{u}{CR} = \frac{du}{dt}$$

Separate the variables gives

$$-\frac{dt}{CR} = \frac{du}{u}$$

Integrating both sides will introduce an arbitrary constant and the result is

$$-\frac{t}{CR} + D = \ln(u)$$

Back substitute for u

$$-\frac{t}{CR} + D = \ln(T - T_c) \tag{2}$$

Exponentiate each side

$$e^{-t/CR+D} = T - T_c$$

Simplify

$$e^{-t/CR} \cdot e^{D} = T - T_c$$

The temperature at any particular time is

$$T = T_c + Ge^{-t/CR}$$

where $G = e^D$. This is the solution to Newton's law of cooling. This looks similar to decay problems with a time constant of CR. At $t = 0$, $T = T_0$, then

$$T_0 = T_c + Ge^{-0/CR} = T_c + G$$

Solving for G gives

$$G = T_0 - T_c = 20\ {}^\circ C - (-15^\circ C) = 35\ {}^\circ C$$

Since $G = e^D$, then

$$\ln(G) = \ln(e^D) = D$$

Therefore,

$$D = \ln(35\ {}^\circ C)$$

Using equation 2 solve for time when $T = 0\ {}^\circ C$.

$$-\frac{t}{CR} = \ln(T - T_c) - D$$

$$t = -CR\left[\ln(T - T_c) - D\right]$$

Substitute in the appropriate values gives

$$t = -(6.5 \times 10^6\ J/K)(0.00667\ K/W)\left[\ln(0\ ^\circ C - (-15\ ^\circ C) - \ln(35\ ^\circ C)\right]$$

$$t = -(6.5 \times 10^6\ J/K)(0.00667\ K/W)\left[\ln(15\ ^\circ C) - \ln(35\ ^\circ C)\right] \tag{3}$$

$$t = 36,700\ s = 10.2\ hours$$

Some of you are bothered by the fact that I took the logarithm of a number with units, and you should be. But look closely at the brackets on the right side of equation 3.
Using the law of logarithms we get

$$\ln(15\ ^\circ C) - \ln(35\ ^\circ C) = \ln\left(\frac{15\ ^\circ C}{35\ ^\circ C}\right) = \ln(0.429)$$

This number is unitless and when substituted into equation 3 gives

$$t = -(6.5 \times 10^6\ J/K)(0.00667\ K/W)\ln(0.429) = 36,700\ s$$

which is the same answer.