Chapter 35 Problem 53 †

Given
$U = \frac{1}{2}m\omega^2 x^2$

Solution
a) What is Schrodinger's equation for the harmonic oscillator.

Schrodinger's equation is

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+U(x,y,z)\psi = E\psi$$

In one dimension it becomes

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}\right)+U\psi = E\psi$$

Substituting in the potential energy function gives

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}\right)+\tfrac{1}{2}m\omega^2 x^2\psi = E\psi$$

b) Show that $\psi = A_0 e^{-\alpha^2 x^2/2}$ satisfies this equation where $\alpha^2 = m\omega/\hbar$.

Take the first derivative of this function with respect to x.

$$\frac{\partial\psi}{\partial x} = \frac{\partial\left(A_0 e^{-\alpha^2 x^2/2}\right)}{\partial x} = A_0 e^{-\alpha^2 x^2/2}(-\alpha^2 2x/2)$$

$$\frac{\partial\psi}{\partial x} = A_0 e^{-\alpha^2 x^2/2}(-\alpha^2 x) = \psi(-\alpha^2 x)$$

Take the second derivative

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial(\partial\psi/\partial x)}{\partial x} = \frac{\partial\left(A_0 e^{-\alpha^2 x^2/2}(-\alpha^2 x)\right)}{\partial x}$$

$$\frac{\partial^2\psi}{\partial x^2} = A_0 e^{-\alpha^2 x^2/2}(-\alpha^2) + A_0 e^{-\alpha^2 x^2/2}(-\alpha^2 x)^2$$

$$\frac{\partial^2\psi}{\partial x^2} = A_0 e^{-\alpha^2 x^2/2}\left[(-\alpha^2)+(-\alpha^2 x)^2\right] = \left[(-\alpha^2)+(-\alpha^2 x)^2\right]\psi$$

Now substitute this into Schrodinger's equation

$$-\frac{\hbar^2}{2m}\left(\left[(-\alpha^2)+(-\alpha^2 x)^2\right]\psi\right)+\left(\tfrac{1}{2}m\omega^2 x^2\right)\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\left(\left[(-\alpha^2)+(-\alpha^2 x)^2\right]\psi\right)+\left(\tfrac{1}{2}m\omega^2 x^2\right)\psi = E\psi$$

Factor out ψ, energy becomes

$$E = -\frac{\hbar^2}{2m}\left(\left[(-\alpha^2)+(-\alpha^2 x)^2\right]\right)+\left(\tfrac{1}{2}m\omega^2 x^2\right)$$

† Problem from Essential University Physics, Wolfson

Since $\alpha^2 = m\omega/\hbar$, then

$$E = -\frac{\hbar^2}{2m}\left(\left[(-m\omega/\hbar) + (-m\omega x/\hbar)^2\right]\right) + \left(\tfrac{1}{2}m\omega^2 x^2\right)$$

$$E = -\frac{\hbar^2}{2m}\left(\frac{-m\omega}{\hbar} + \frac{m^2\omega^2 x^2}{\hbar^2}\right) + \left(\tfrac{1}{2}m\omega^2 x^2\right)$$

$$E = \left(\frac{\hbar\omega}{2} - \tfrac{1}{2}m\omega^2 x^2\right) + \left(\tfrac{1}{2}m\omega^2 x^2\right) = \tfrac{1}{2}\hbar\omega$$

This is the ground state of the harmonic oscillator.

c) Normalize the function and find the value of A_0.

The solution to the harmonic oscillator is a bell curve and normalization involves a subtitle process. Write out the normalization process for ψ

$$1 = \int_{-\infty}^{\infty} \psi^2 dx$$

$$1 = \int_{-\infty}^{\infty} \left(A_0 e^{-\alpha^2 x^2/2}\right)^2 dx = A_0^2 \int_{-\infty}^{\infty} \left(e^{-\alpha^2 x^2}\right) dx = 1$$

Now let's square the integration portion and take the square root.

$$1 = A_0^2 \sqrt{\left[\int_{-\infty}^{\infty} \left(e^{-\alpha^2 x^2}\right) dx\right]^2}$$

Write out each term of the square based on a different variable

$$1 = A_0^2 \sqrt{\left[\int_{-\infty}^{\infty} \left(e^{-\alpha^2 x^2}\right) dx\right]\left[\int_{-\infty}^{\infty} \left(e^{-\alpha^2 y^2}\right) dy\right]}$$

Since each variable is independent of the other the two individual integrals can be treated as a double integral.

$$1 = A_0^2 \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\alpha^2 x^2} e^{-\alpha^2 y^2} dxdy} = A_0^2 \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\alpha^2 (x^2+y^2)} dxdy}$$

Notice this double integral is over all space in Cartesian coordinates. Change to polar coordinates and $dxdy$ becomes $rdrd\theta$ and $x^2 + y^2$ becomes r^2. Also the limits for θ go from 0 to 2π and the limits for r go from 0 to ∞.

$$1 = A_0^2 \sqrt{\int_0^{2\pi}\int_0^{\infty} e^{-\alpha^2 r^2} rdrd\theta}$$

Make a u-substitution on r where $u = -\alpha^2 r^2$ and $du = -2\alpha^2 rdr$. The limits now go from 0 to $-\infty$.

$$1 = A_0^2 \sqrt{\int_0^{2\pi}\int_0^{-\infty} \frac{e^u}{-2\alpha^2} dud\theta}$$

Solving the integral gives

$$1 = A_0^2 \sqrt{\frac{1}{-2\alpha^2}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{-\infty} e^u du\right)} = A_0^2 \sqrt{\frac{1}{-2\alpha^2}\left(2\pi - 0\right)\left(e^{-\infty} - e^0\right)}$$

$$1 = A_0^2 \sqrt{\frac{1}{-2\alpha^2} (2\pi) (0 - 1)} = A_0^2 \sqrt{\frac{2\pi}{2\alpha^2}} = A_0^2 \sqrt{\frac{\pi}{\alpha^2}}$$

Solving for A_0 gives

$$A_0 = \sqrt[4]{\frac{\alpha^2}{\pi}}$$

Since $\alpha^2 = m\omega/\hbar$, then

$$A_0 = \sqrt[4]{\frac{m\omega}{\pi\hbar}}$$