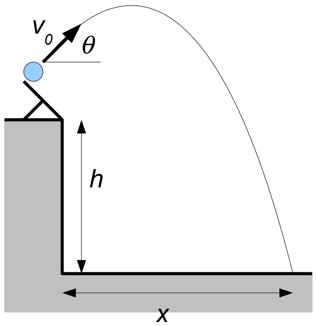
Chapter 3 Problem 78 †



Given
$h = 75\ m$
$v_0 = 36\ m/s$
$g = 9.80\ m/s^2$

Solution
Find the angle that gives the maximum range.

Assuming there is no air drag, the only force acting on the stone after the launch is gravity. Since gravity only acts in the vertical direction, the equation of motion in the vertical is

$$y = y_0 + v_{y0}t + \tfrac{1}{2}a_y t^2$$

Setting the origin at the base of the cliff and having the upward direction as positive, the initial position of the rock is $y_0 = h$ and the initial vertical velocity is $y_{y0} = v_0 \sin\theta$. The acceleration in the vertical direction is downward, $a_y = -g$. The final height of the stone is $y = 0$. Substituting into the vertical equation gives

$$0 = h + v_0 \sin\theta t - \tfrac{1}{2}gt^2 \tag{1}$$

In the horizontal direction the equation of motion is

$$x = x_0 + v_{x0}t + \tfrac{1}{2}a_x t^2$$

Since there is no acceleration in the horizontal direction, $a_x = 0$. With the origin at the base of the cliff, $x_0 = 0$. Finally the initial horizontal velocity is $v_{x0} = v_0 \cos\theta$. Putting these together gives

$$x = v_0 \cos\theta t \tag{2}$$

Solving equation 1 for time involves the quadratic formula. The two solutions to this equation are

$$t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-v_0 \sin\theta \pm \sqrt{(v_0 \sin\theta)^2 - 4(-g/2)(h)}}{2(-g/2)}$$

Simplifying gives

$$t = \frac{v_0 \sin\theta \mp \sqrt{(v_0 \sin\theta)^2 + 2gh}}{g}$$

†Problem from Essential University Physics, Wolfson

The solution that gives a positive value for time will be the sum of the terms rather than the difference. Substituting this into the horizontal motion equation (2) gives a range of

$$x = v_0 \cos\theta \left(\frac{v_0 \sin\theta + \sqrt{(v_0 \sin\theta)^2 + 2gh}}{g} \right)$$

or

$$x = \frac{v_0 \cos\theta}{g} \left(v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta + 2gh} \right) \tag{3}$$

To maximize range with respect to angle, the derivative of x with respect to θ must be set to zero.

$$0 = \frac{dx}{d\theta} = \frac{-v_0 \sin\theta}{g} \left(v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta + 2gh} \right) + \frac{v_0 \cos\theta}{g} \left(v_0 \cos\theta + \frac{2v_0^2 \sin\theta \cos\theta}{2\sqrt{v_0^2 \sin^2\theta + 2gh}} \right)$$

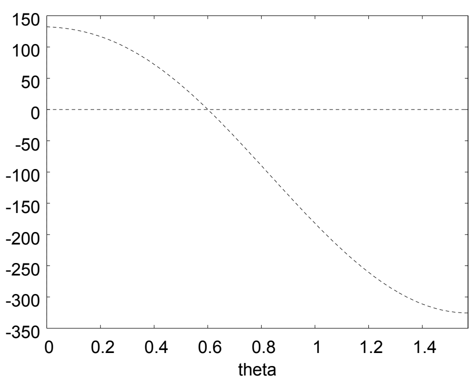
This relationship is "ugly" and should be solved numerically by a computer. Substituting in values for h, v_0, and g gives

$$0 = \frac{dx}{d\theta} = \frac{-36 \sin\theta}{9.8} \left(36 \sin\theta + \sqrt{36^2 \sin^2\theta + 2(9.8)(75)} \right) + \frac{36 \cos\theta}{9.8} \left(36 \cos\theta + \frac{36^2 \sin\theta \cos\theta}{\sqrt{36^2 \sin^2\theta + 2(9.8)(75)}} \right)$$

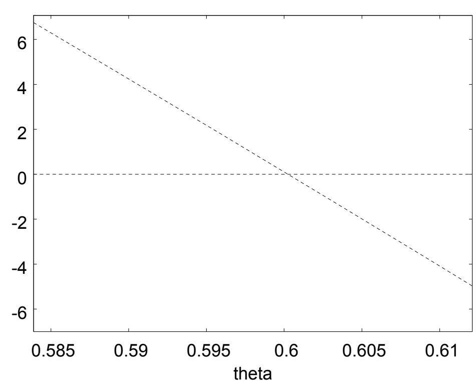
Simplifying gives

$$0 = \frac{dx}{d\theta} = -3.67 \sin\theta \left(36 \sin\theta + \sqrt{1296 \sin^2\theta + 1470} \right) + 3.67 \cos\theta \left(36 \cos\theta + \frac{1296 \sin\theta \cos\theta}{\sqrt{1296 \sin^2\theta + 1470}} \right)$$

Plotting this function in an algebraic software package like Maxima or Maple gives the following plots.



Zooming in gives



This function crosses zero at $\theta = 0.60\ rad$. Therefore, the launch angle for maximum range is

$$\theta = 0.60\ rad \left(\frac{360°}{2\pi\ rad} \right) = 34°$$