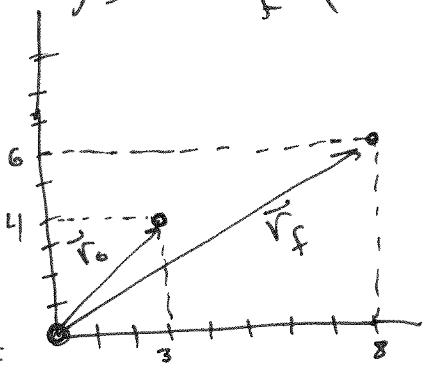
Chapter 7 Problem 42

$$\vec{F}(x,y) = (50\,N\cdot m^2)\frac{(x\hat{i} + y\hat{j})}{(x^2+y^2)^{3/2}}$$

Calculate the work done on the particle as it moves on a straight line from $\vec{r}_o = \{3\hat{i} + 4\hat{j}\}\,m$ to $\vec{r}_f = \{8\hat{i} + 6\hat{j}\}\,m$



Since we are following a straight line, find the equation of the line

$$\text{slope} = m = \frac{\Delta y}{\Delta x} = \frac{Y_f - Y_o}{X_f - X_o} = \frac{6-4}{8-3} = \frac{2}{5} = 0.400$$

The y-intercept is

$$y = mx + b \rightarrow b = y - mx = 4 - \frac{2}{5}(3) = \frac{20}{5} - \frac{6}{5}$$

$$b = \frac{14}{5} = 2.80$$

$$y = 0.40x + 2.80 \quad (\#1)$$

The infinitesmal displacement vector is $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

we don't want to do a 2-D integral (from Calculus III), so use the equation of the line to reduce it to a 1-D integral

$$\text{from } y = 0.40x + 2.80 \rightarrow \frac{dy}{dx} = 0.40 \quad \therefore \; dy = 0.40dx \quad (\#2)$$

Now let's set up the integral

$$W = \int_{\vec{r}_o}^{\vec{r}_f} \vec{F}(x,y)\cdot d\vec{r} = \int_{\vec{r}_o}^{\vec{r}_f} (50\,Nm^2)\left[\frac{x\hat{i} + y\hat{j}}{(x^2+y^2)^{3/2}}\right]\cdot\left[dx\,\hat{i} + dy\,\hat{j}\right]$$

Replace dy with $0.40dx$ (eq #2) and y with $0.40x + 2.80$ (eq #1)

$$W = \int_{x_o}^{x_f} (50\,Nm^2)\left[\frac{x\hat{i} + (0.40x + 2.80)\hat{j}}{\left[x^2 + (0.40x + 2.80)^2\right]^{3/2}}\right]\cdot\left[dx\,\hat{i} + 0.40dx\,\hat{i}\right]$$

Perform the dot product

$$W = \int_{x_o}^{x_f} (50\,Nm^2)\left[\frac{x\,dx + (0.40x + 2.80)(0.40\,dx)}{\left[x^2 + (0.40x)^2 + 2(0.40x)(2.80) + (2.80)^2\right]^{3/2}}\right]$$

$$W = \int_{x_o}^{x_f} (50\,Nm^2) \frac{[x + 0.16x + 1.12]\,dx}{[1.16x^2 + 2.24x + 7.84]^{3/2}}$$

$$= \int_{x=3}^{x=8} (50\,Nm^2) \frac{(1.16x + 1.12)\,dx}{(1.16x^2 + 2.24x + 7.84)^{3/2}}$$

let's do a "U"-substitution

$$u = 1.16x^2 + 2.24x + 7.84 \qquad \#3$$

$$\frac{du}{dx} = 2(1.16)x + 2.24 = 2.32x + 2.24$$

$$du = (2.32x + 2.24)\,dx$$

Divide both sides by 2

$$\frac{du}{2} = (1.16x + 1.12)\,dx$$

This now makes the integral

$$W = \int_{x=3}^{x=8} (50\,Nm^2) \frac{\frac{du}{2}}{u^{3/2}} = \int_{x=3}^{x=8} 25\,Nm^2\, u^{-3/2}\,du$$

Note: will drop the units from here on out. Before the U-sub, the work comes to N·m or J

Now change the limits using equation #3

$x_o = 3 \qquad u_o = 1.16(3)^2 + 2.24(3) + 7.84$
$= 25$

$x_f = 8 \qquad u_f = 1.16(8)^2 + 2.24(8) + 7.84$
$= 100$

$$W = 25 \int_{25}^{100} u^{-3/2}\,du = 25 \left.\frac{u^{-1/2}}{-1/2}\right|_{25}^{100} = \left.\frac{-50}{u^{1/2}}\right|_{25}^{100}$$

$$= -50\left[\frac{1}{(100)^{1/2}} - \frac{1}{(25)^{1/2}}\right] = -50\left(\frac{1}{10} - \frac{1}{5}\right) = -50\left[\frac{1-2}{10}\right]$$

$$W = \boxed{5.0\,J}$$